Q12 * Mathematik * Differenzieren und Integrieren

1. Bestätigen Sie durch Rechnung die angegebenen Ableitungsfunktionen!

a) $f(x) = (2x-3)^3 \quad \Rightarrow \quad f'(x) = 6\cdot(2x-3)^2$

b) $f(x) = 2\cdot\sqrt{x^2+3} \quad \Rightarrow \quad f'(x) = \dfrac{2x}{\sqrt{x^2+3}}$

c) $f(x) = (x^2+3x)\cdot(x^2-1) \quad \Rightarrow \quad f'(x) = 4x^3+9x^2-2x-3$

d) $f(x) = \dfrac{2x}{x^2+1} \quad \Rightarrow \quad f'(x) = \dfrac{2\cdot(1-x^2)}{(x^2+1)^2}$

e) $f(x) = \dfrac{1+x^2}{2x} \quad \Rightarrow \quad f'(x) = \dfrac{x^2-1}{2x^2} = \dfrac{1}{2} - \dfrac{1}{2x^2}$

f) $f(x) = \dfrac{\sqrt{2x+1}}{x^2+1} \quad \Rightarrow \quad f'(x) = \dfrac{1-2x-3x^2}{(x^2+1)^2\cdot\sqrt{2x+1}}$

g) $f(x) = \dfrac{3x-1}{\sqrt{x^2+1}} \quad \Rightarrow \quad f'(x) = \dfrac{3+x}{(x^2+1)\cdot\sqrt{x^2+1}}$

h) $f(x) = \dfrac{x^2-1}{x^2+1} \quad \Rightarrow \quad f'(x) = \dfrac{4x}{(x^2+1)^2}$

i) $f(x) = -x + x\cdot\ln(x) \quad \Rightarrow \quad f'(x) = \ln(x)$

j) $f(x) = 2x\cdot\ln(1+x^2) \quad \Rightarrow \quad f'(x) = \dfrac{4x^2}{x^2+1} + 2\cdot\ln(1+x^2)$

k) $f(x) = \ln\left(\dfrac{2x}{x^2+1}\right) \quad \Rightarrow \quad f'(x) = \dfrac{1-x^2}{x\cdot(x^2+1)}$

l) $f(x) = \ln\left(\sqrt{x^2+1}\right) \quad \Rightarrow \quad f'(x) = \dfrac{x}{x^2+1}$

m) $f(x) = \ln\left(x\cdot\sqrt{x^2+1}\right) \quad \Rightarrow \quad f'(x) = \dfrac{2x^2+1}{x\cdot(x^2+1)}$

n) $f(x) = 2x\cdot e^{x+1} \quad \Rightarrow \quad f'(x) = 2\cdot(x+1)\cdot e^{x+1}$

o) $f(x) = (x^2-x)\cdot e^{2x} \quad \Rightarrow \quad f'(x) = (2x^2-1)\cdot e^{2x}$

p) $f(x) = \dfrac{2-e^x}{1+e^x} \quad \Rightarrow \quad f'(x) = \dfrac{-3\cdot e^x}{(1+e^x)^2}$

q) $f(x) = \dfrac{e^{2x+1}}{e^{2x}+1} \quad \Rightarrow \quad f'(x) = \dfrac{2e^{2x+1}}{(e^{2x}+1)^2}$

2. Bestimmen Sie (mit Hilfe von Aufgabe 1) den Wert der folgenden bestimmten Integrale!

a) $\displaystyle\int_1^3 \frac{2x}{\sqrt{x^2+3}}\,dx$ b) $\displaystyle\int_{-2}^{-1} \frac{x^2-1}{2x^2}\,dx$ c) $\displaystyle\int_0^1 \frac{3\cdot e^x}{(1+e^x)^2}\,dx$

d) $\displaystyle\int_1^e \ln(x)\,dx$ e) $\displaystyle\int_0^1 (2x^2-1)\cdot e^{2x}\,dx$

Q12 * Mathematik * Differenzieren und Integrieren * Lösungen

1.

a) $f(x) = (2x-3)^3 \quad\Rightarrow\quad f'(x) = 3\cdot(2x-3)^2\cdot 2 = 6\cdot(2x-3)^2$

b) $f(x) = 2\cdot\sqrt{x^2+3} \quad\Rightarrow\quad f'(x) = 2\cdot\frac{2x}{2\cdot\sqrt{x^2+3}} = \frac{2x}{\sqrt{x^2+3}}$

c) $f(x) = (x^2+3x)\cdot(x^2-1) \quad\Rightarrow\quad f(x) = x^4+3x^3-x^2-3x \quad\Rightarrow\quad f'(x) = 4x^3+9x^2-2x-3$

d) $f(x) = \frac{2x}{x^2+1} \quad\Rightarrow\quad f'(x) = \frac{2\cdot(x^2+1)-2x\cdot 2x}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{(x^2+1)^2} = \frac{2-2x^2}{(x^2+1)^2} = \frac{2\cdot(1-x^2)}{(x^2+1)^2}$

e) $f(x) = \frac{1+x^2}{2x} \quad\Rightarrow\quad f'(x) = \frac{2x\cdot 2x-(1+x^2)\cdot 2}{4x^2} = \frac{2x^2-2}{4x^2} = \frac{x^2-1}{2x^2} = \frac{1}{2}-\frac{1}{2x^2}$

f) $f(x) = \frac{\sqrt{2x+1}}{x^2+1} \quad\Rightarrow\quad f'(x) = \frac{(x^2+1)\cdot\frac{2}{2\cdot\sqrt{2x+1}} - \sqrt{2x+1}\cdot 2x}{(x^2+1)^2} = \frac{(x^2+1)-(2x+1)\cdot 2x}{(x^2+1)^2\cdot\sqrt{2x+1}} =$

$\frac{x^2+1-4x^2-2x}{(x^2+1)^2\cdot\sqrt{2x+1}} = \frac{1-2x-3x^2}{(x^2+1)^2\cdot\sqrt{2x+1}}$

g) $f(x) = \frac{3x-1}{\sqrt{x^2+1}} \quad\Rightarrow\quad f'(x) = \frac{3\cdot\sqrt{x^2+1}-(3x-1)\cdot\frac{2x}{2\cdot\sqrt{x^2+1}}}{x^2+1} = \frac{3\cdot(x^2+1)-(3x-1)\cdot x}{(x^2+1)\cdot\sqrt{x^2+1}} =$

$\frac{3x^2+3-3x^2+x}{(x^2+1)\cdot\sqrt{x^2+1}} = \frac{3+x}{(x^2+1)\cdot\sqrt{x^2+1}}$

h) $f(x) = \frac{x^2-1}{x^2+1} \quad\Rightarrow\quad f'(x) = \frac{2x\cdot(x^2+1)-(x^2-1)\cdot 2x}{(x^2+1)^2} = \frac{2x^3+2x-2x^3+2x}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$

i) $f(x) = -x+x\cdot\ln(x) \quad\Rightarrow\quad f'(x) = -1+1\cdot\ln(x)+x\cdot\frac{1}{x} = -1+\ln(x)+1 = \ln(x)$

j) $f(x) = 2x\cdot\ln(1+x^2) \quad\Rightarrow\quad f'(x) = 2\cdot\ln(1+x^2)+2x\cdot\frac{1\cdot 2x}{x^2+1} = \frac{4x^2}{x^2+1}+2\cdot\ln(1+x^2)$

k) $f(x) = \ln\left(\frac{2x}{x^2+1}\right) \quad\Rightarrow\quad f'(x) = \frac{x^2+1}{2x}\cdot\frac{(x^2+1)\cdot 2-2x\cdot 2x}{(x^2+1)^2} = \frac{2x^2+2-4x^2}{2x\cdot(x^2+1)} = \frac{2-2x^2}{2x\cdot(x^2+1)} = \frac{1-x^2}{x\cdot(x^2+1)}$

l) $f(x) = \ln\left(\sqrt{x^2+1}\right) \quad\Rightarrow\quad f'(x) = \frac{1}{\sqrt{x^2+1}}\cdot\frac{2x}{2\cdot\sqrt{x^2+1}} = \frac{x}{x^2+1}$

m) $f(x) = \ln\left(x\cdot\sqrt{x^2+1}\right) \quad\Rightarrow\quad f'(x) = \frac{1}{x\cdot\sqrt{x^2+1}}\cdot\left(1\cdot\sqrt{x^2+1}+x\cdot\frac{2x}{2\cdot\sqrt{x^2+1}}\right) =$

$= \frac{1}{x\cdot\sqrt{x^2+1}}\cdot\frac{(x^2+1)+x^2}{\sqrt{x^2+1}} = \frac{2x^2+1}{x\cdot(x^2+1)}$

n) $f(x) = 2x\cdot e^{x+1} \quad\Rightarrow\quad f'(x) = 2\cdot e^{x+1}+2x\cdot e^{x+1} = (2+2x)\cdot e^{x+1} = 2\cdot(x+1)\cdot e^{x+1}$

o) $f(x) = (x^2-x)\cdot e^{2x} \quad\Rightarrow\quad f'(x) = (2x-1)\cdot e^{2x}+(x^2-x)\cdot e^{2x}\cdot 2 = (2x-1+2x^2-2x)\cdot e^{2x} = (2x^2-1)\cdot e^{2x}$

p) $f(x) = \frac{2-e^x}{1+e^x} \quad\Rightarrow\quad f'(x) = \frac{(1+e^x)\cdot(-e^x)-(2-e^x)\cdot e^x}{(1+e^x)^2} = \frac{-e^x-e^{2x}-2e^x+e^{2x}}{(1+e^x)^2} = \frac{-3\cdot e^x}{(1+e^x)^2}$

q) $f(x) = \frac{e^{2x+1}}{e^{2x}+1} \quad\Rightarrow\quad f'(x) = \frac{(e^{2x}+1)\cdot e^{2x+1}\cdot 2-e^{2x+1}\cdot e^{2x}\cdot 2}{(e^{2x}+1)^2} = \frac{2e^{4x+1}+2e^{2x+1}-2e^{4x+1}}{(e^{2x}+1)^2} = \frac{2e^{2x+1}}{(e^{2x}+1)}$

2.

a) $$\int_1^3 \frac{2x}{\sqrt{x^2+3}}\,dx \underset{1b)}{=} \left[\,2\cdot\sqrt{x^2+3}\,\right]_1^3 = 2\cdot\sqrt{9+3} - 2\cdot\sqrt{1+3} = 4\cdot\sqrt{3} - 4 = 4\cdot(\sqrt{3}-1) \approx 2{,}93$$

b) $$\int_{-2}^{-1} \frac{x^2-1}{2x^2}\,dx \underset{1e)}{=} \left[\frac{1+x^2}{2x}\right]_{-2}^{-1} = \frac{1+1}{-2} - \frac{1+4}{-4} = -1 + \frac{5}{4} = \frac{1}{4} \quad \text{oder}$$

$$\int_{-2}^{-1} \frac{x^2-1}{2x^2}\,dx = \int_{-2}^{-1} \frac{1}{2} - \frac{1}{2x^2}\,dx = \left[\frac{x}{2} + \frac{1}{2x}\right]_{-2}^{-1} = \frac{-1}{2} + \frac{1}{-2} - \left(\frac{-2}{2} + \frac{1}{-4}\right) = -1 + \frac{5}{4} = \frac{1}{4}$$

c) $$\int_0^1 \frac{3\cdot e^x}{(1+e^x)^2}\,dx \underset{1p)}{=} -\left[\frac{2-e^x}{1+e^x}\right]_0^1 = -\left(\frac{2-e}{1+e}\right) + \frac{2-1}{1+1} = \frac{e-2}{e+1} + \frac{1}{2} \approx 0{,}693$$

d) $$\int_1^e \ln(x)\,dx \underset{1i)}{=} \left[-x + x\cdot\ln(x)\right]_1^e = -e + e\cdot\ln e - (-1 + 1\cdot\ln(1)) = 0 - (-1 + 1\cdot 0) = 1$$

e) $$\int_0^1 (2x^2-1)\cdot e^{2x}\,dx \underset{1o)}{=} \left[(x^2-x)\cdot e^{2x}\right]_0^1 = (1-1)\cdot e^2 - (0-0)\cdot e^0 = 0$$